Understanding Convolutional Neural Networks

David Stutz

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Motivation

Convolutional networks represent specialized networks for application in computer vision:

- they accept images as raw input (preserving spatial information),
- and build up (learn) a hierarchy of features (no hand-crafted features necessary).

Problem: Internal workings of convolutional networks not well understood ...

- Unsatisfactory state for evaluation and research!

Idea: Visualize feature activations within the network ...

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Multilayer Perceptrons

A multilayer perceptron represents an adaptable model $y(\cdot, w)$ able to map D-dimensional input to C-dimensional output:

$$y(\cdot, w) : \mathbb{R}^D \to \mathbb{R}^C, x \mapsto y(x, w) = \begin{pmatrix} y_1(x, w) \\ \vdots \\ y_C(x, w) \end{pmatrix}. \qquad (1)$$

In general, a $(L + 1)$-layer perceptron consists of $(L + 1)$ layers, each layer l computing linear combinations of the previous layer $(l - 1)$ (or the input).

Multilayer Perceptrons – First Layer

On input $x \in \mathbb{R}^D$, layer $l = 1$ computes a vector $y^{(1)} := (y_1^{(1)}, \ldots, y_{m^{(1)}}^{(1)})$ where

$$y_i^{(1)} = f\left(z_i^{(1)}\right) \quad \text{with } z_i^{(1)} = \sum_{j=1}^{D} w_{i,j}^{(1)} x_j + w_{i,0}^{(1)}. \tag{2}$$

i^{th} component is called “unit i”

where f is called activation function and $w_{i,j}^{(1)}$ are adjustable weights.

Multilayer Perceptrons – First Layer

What does this mean?

Layer $l = 1$ computes linear combinations of the input and applies an (non-linear) activation function ...

The first layer can be interpreted as generalized linear model:

$$y_i^{(1)} = f\left(\left(w_i^{(1)}\right)^T x + w_{i,0}^{(1)}\right). \quad (3)$$

Idea: Recursively apply L additional layers on the output $y^{(1)}$ of the first layer.

Multilayer Perceptrons – Further Layers

In general, layer l computes a vector $y^{(l)} := (y_1^{(l)}, \ldots, y_{m^{(l)}}^{(l)})$ as follows:

$$y_i^{(l)} = f\left(z_i^{(l)}\right) \quad \text{with } z_i^{(l)} = \sum_{j=1}^{m^{(l-1)}} w_{i,j}^{(l)} y_j^{(l-1)} + w_{i,0}^{(l)}. \tag{4}$$

Thus, layer l computes linear combinations of layer $(l-1)$ and applies an activation function ...

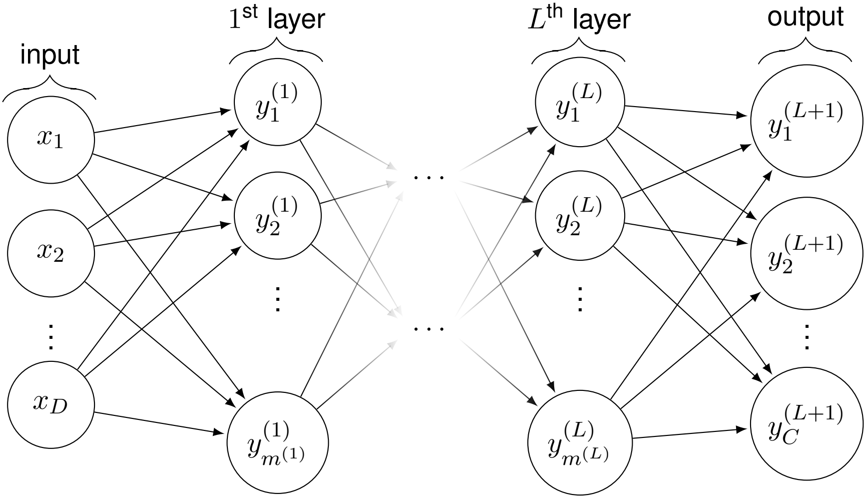
Multilayer Perceptrons – Output Layer

Layer $(L + 1)$ is called output layer because it computes the output of the multilayer perceptron:

$$y(x, w) = \begin{pmatrix} y_1(x, w) \\ \vdots \\ y_C(x, w) \end{pmatrix} := \begin{pmatrix} y_1^{(L+1)} \\ \vdots \\ y_C^{(L+1)} \end{pmatrix} = y^{(L+1)} \quad (5)$$

where $C = m^{(L+1)}$ is the number of output dimensions.

Network Graph



Activation Functions – Notions

How to choose the activation function f in each layer?

- Non-linear activation functions will increase the expressive power: Multilayer perceptrons with $L + 1 \geq 2$ are universal approximators [HSW89]!
- Depending on the application: For classification we may want to interpret the output as posterior probabilities:

$$y_i(x, w) \overset{!}{=} p(c = i|x) \quad (6)$$

where c denotes the random variable for the class.

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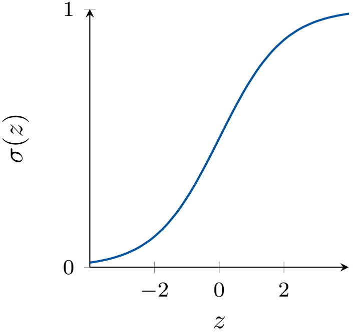
Activation Functions

Usually the activation function is chosen to be the logistic sigmoid:

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$



which is non-linear, monotonic and differentiable.

Activation Functions

Alternatively, the hyperbolic tangent is used frequently:

$$\tanh(z). \tag{7}$$

For classification with $C > 1$ classes, layer $(L + 1)$ uses the softmax activation function:

$$y_i^{(L+1)} = \sigma(z^{(L+1)}, i) = \frac{\exp(z_i^{(L+1)})}{\sum_{k=1}^{C} \exp(z_k^{(L+1)})}. \tag{8}$$

Then, the output can be interpreted as posterior probabilities.

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Network Training – Notions

By now, we have a general model $y(\cdot, w)$ depending on W weights.

Idea: Learn the weights to perform

- regression,
- or classification.

We focus on classification.

Network Training – Training Set

C classes:

1-of-C coding scheme

Given a training set

$$U_S = \{(x_n, t_n) : 1 \leq n \leq N\}, \tag{9}$$

learn the mapping represented by U_S ...

by minimizing the squared error

$$E(w) = \sum_{n=1}^{N} E_n(w) = \sum_{n=1}^{N} \sum_{i=1}^{C} (y_i(x_n, w) - t_{n,i})^2 \tag{10}$$

using iterative optimization.

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Training Protocols

We distinguish ...

Stochastic Training A training sample (x_n, t_n) is chosen at random, and the weights w are updated to minimize $E_n(w)$.

Batch and Mini-Batch Training A set $M \subseteq \{1, \ldots, N\}$ of training samples is chosen and the weights w are updated based on the cumulative error $E_M(w) = \sum_{n \in M} E_n(w)$.

Of course, online training is possible, as well.

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Iterative Optimization

Problem: How to minimize $E_n(w)$ (stochastic training)?

- $E_n(w)$ may be highly non-linear with many poor local minima.

Framework for iterative optimization: Let ...

- $w[0]$ be an initial guess for the weights (several initialization techniques are available),
- and $w[t]$ be the weights at iteration t.

In iteration $[t + 1]$, choose a weight update $\Delta w[t]$ and set

$$w[t + 1] = w[t] + \Delta w[t] \tag{11}$$

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Gradient Descent

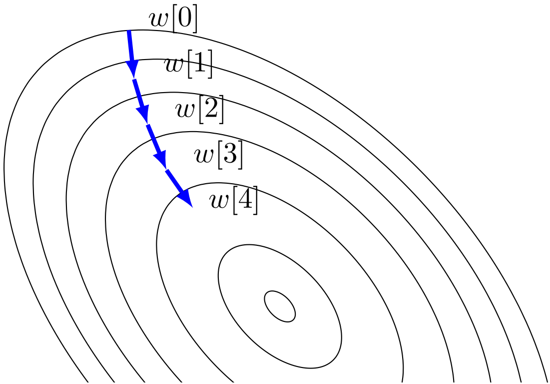
Remember:

Gradient descent minimizes the error $E_n(w)$ by taking steps in the direction of the negative gradient:

$$\Delta w[t] = -\gamma \frac{\partial E_n}{\partial w[t]} \tag{12}$$

where γ defines the step size.

Gradient Descent – Visualization



Error Backpropagation

Problem: How to evaluate $\frac{\partial E_n}{\partial w[t]}$ in iteration $[t+1]$?

- "Error Backpropagation" allows to evaluate $\frac{\partial E_n}{\partial w[t]}$ in $\mathcal{O}(W)$!

Further details ...

- See the original paper "Learning Representations by Back-Propagating Errors," by Rumelhart et al. [RHW86].

Deep Learning

Multilayer perceptrons are called deep if they have more than three layers: $L + 1 > 3$.

Motivation: Lower layers can automatically learn a hierarchy of features or a suitable dimensionality reduction.

- No hand-crafted features necessary anymore!

However, training deep neural networks is considered very difficult!

- Error measure represents a highly non-convex, “potentially intractable” [EMB$^+$09] optimization problem.

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Approaches to Deep Learning

Possible approaches:

- Different activation functions offer faster learning, for example

$$\max(0, z) \quad \text{or} \quad |\tanh(z)|; \tag{13}$$

- unsupervised pre-training can be done layer-wise;
- ...

Further details ...

- See “Learning Deep Architectures for AI,” by Y. Bengio [Ben09] for a detailed discussion of state-of-the-art approaches to deep learning.

Summary

The multilayer perceptron represents a standard model of neural networks. They ...

- allow to taylor the architecture (layers, activation functions) to the problem;
- can be trained using gradient descent and error backpropagation;
- can be used for learning feature hierarchies (deep learning).

Deep learning is considered difficult.

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Convolutional Networks

Idea: Allow raw image input while preserving the spatial relationship between pixels.

Tool: Discrete convolution of image I with filter $K \in \mathbb{R}^{2h_1+1 \times 2h_2+1}$ is defined as

$$(I * K)_{r,s} = \sum_{u=-h_1}^{h_1} \sum_{v=-h_2}^{h_2} K_{u,v} I_{r+u,s+v} \qquad (14)$$

where the filter K is given by

$$K = \begin{pmatrix} K_{-h_1,-h_2} & \dots & K_{-h_1,h_2} \\ \vdots & K_{0,0} & \vdots \\ K_{h_1,-h_2} & \dots & K_{h_1,h_2} \end{pmatrix}. \qquad (15)$$

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Convolutional Networks – Architectures

Original Convolutional Network [LBD+89] aims to build up a feature hierarchy by alternating

applies activation function

subsamples the feature maps

convolutional layer – non-linearity layer – subsampling layer

convolves the image with a set of filters

followed by a multilayer perceptron for classification.

Convolutional Layer – Notions

Central part of convolutional networks: convolutional layer.

- Can handle raw image input.

Idea: Apply a set of learned filters to the image in order to obtain a set of feature maps.

Can be repeated: Apply a different set of filters to the obtained feature maps to get more complex features:

- Generate a hierarchy of feature maps.

Convolutional Layer

Let layer l be a convolutional layer.

Input: $m_1^{(l-1)}$ feature maps $Y_i^{(l-1)}$ of size $m_2^{(l-1)} \times m_3^{(l-1)}$ from the previous layer.

Output: $m_1^{(l)}$ feature maps of size $m_2^{(l)} \times m_3^{(l)}$ given by

layer l

$$Y_i^{(l)} = B_i^{(l)} + \sum_{j=1}^{m_1^{(l-1)}} K_{i,j}^{(l)} * Y_j^{(l-1)} \quad (16)$$

feature map i

where $B_i^{(l)}$ is called bias matrix and $K_{i,j}^{(l)}$ are the filters to be learned.

Convolutional Layer – Notes

Notes:

- The size $m_2^{(l)} \times m_3^{(l)}$ of the output feature maps depends on the definition of discrete convolution (especially how borders are handled).
- The weights $w_{i,j}^{(l)}$ are hidden in the bias matrix $B_i^{(l)}$ and the filters $K_{i,j}^{(l)}$.

Non-Linearity Layer

Let layer l be a non-linearity layer.

Given $m_1^{(l-1)}$ feature maps, a non-linearity layer applies an activation function to all these feature maps:

$$Y_i^{(l)} = f\left(Y_i^{(l-1)}\right) \tag{17}$$

where f operates point-wise.

Usually, f is the hyperbolic tangent.

Layer l computes $m_1^{(l)} = m_1^{(l-1)}$ feature maps unchanged in size ($m_2^{(l)} = m_2^{(l-1)}$, $m_3^{(l)} = m_3^{(l-1)}$).

Subsampling and Pooling Layer

Motivation: Incorporate invariance to noise and distortions.

Idea: Subsample the feature maps of the previous layer.

Let layer l be a subsampling and pooling layer.

Given $m_1^{(l-1)}$ feature maps of size $m_2^{(l-1)} \times m_3^{(l-1)}$, create $m_1^{(l)} = m_1^{(l-1)}$ feature maps of reduced size.

- For example by placing windows at non-overlapping positions within the feature maps and keeping only the maximum activation per window.

Putting it All Together

Remember: A convolutional network alternates

convolutional layer – non-linearity layer – subsampling layer

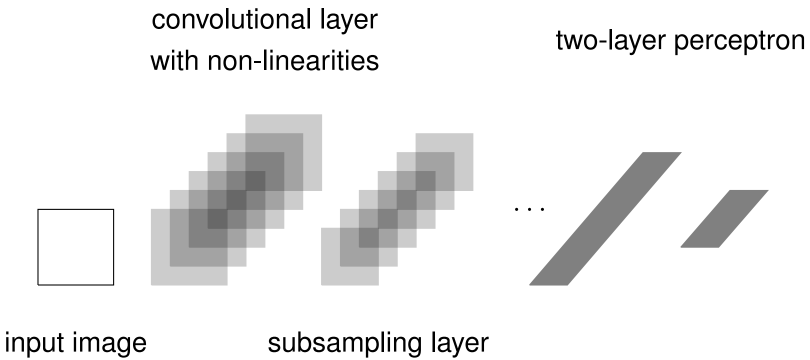
to build up a hierarchy of feature maps...

and uses a multilayer perceptron for classification.

Further details ...

- LeCun et al. [LKF10] and Jarrett et al. [JKRL09] give a review of recent architectures.

Overall Architecture



Additional Layers

Researchers are constantly coming up with additional types of layers ...

Example 1: Let layer l be a rectification layer.

Given feature maps $Y_i^{(l-1)}$ of the previous layer, a rectification layer computes

$$Y_i^{(l)} = \left|Y_i^{(l-1)}\right| \quad (18)$$

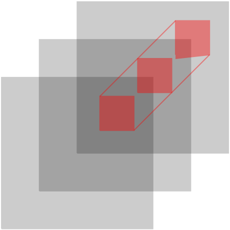
where the absolute value is computed point-wise.

Experiments show that rectification plays an important role to achieve good performance.

Additional Layers (cont'd)

Example 2:

Local contrast normalization layers aim to enforce local competitiveness between adjacent feature maps.



ensure that values are comparable

- There are different implementations available, see Krizhevsky et al. [KSH12] or LeCun et al. [LKF10].

Summary

A basic convolutional network consists of different types of layers:

- convolutional layers;
- non-linearity layers;
- and subsampling layers.

Researchers are constantly thinking about additional types of layers to improve learning and performance.

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Understanding Convolutional Networks

State: Convolutional networks perform well without requiring hand-crafted features.

- But: Learned feature hierarchy not well understood.

Idea: Visualize feature activations of higher convolutional layers ...

- Feature activations after first convolutional layer can be backprojected onto the image plane.

Zeiler et al. [ZF13] propose a visualization technique based on *de*convolutional networks.

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Deconvolutional Networks

*De*convolutional networks aim to build up a feature hierarchy ...

- by convolving the input image by a set of filters – like convolutional networks;
- however, they are fully unsupervised.

Idea: Given an input image (or a set of feature maps), try to reconstruct the input given the filters and their activations.

Basic component: *de*convolutional layer.

Deconvolutional Layer

Let layer l be a *de*convolutional layer.

Given feature maps $Y_i^{(l-1)}$ of the previous layer, try to reconstruct the input using the filters and their activations:

$$Y_i^{(l-1)} \stackrel{!}{=} \sum_{j=1}^{m_1^{(l)}} \left(K_{j,i}^{(l)}\right)^T * Y_j^{(l)}. \quad (19)$$

*De*convolutional layers ...

- are unsupervised by definition;
- need to learn feature activations *and* filters.

Deconvolutional Networks

*De*convolutional networks stack *de*convolutional layers and are fully unsupervised.

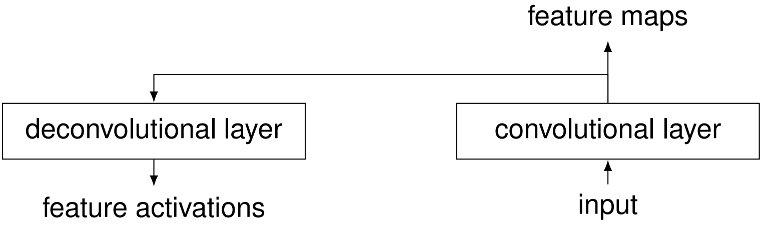
Further details ...

- See “Deconvolutional Networks,” by Zeiler et al. [ZKTF10] for details on how to train deconvolutional networks.

Deconvolutional Layers for Visualization

Here: *De*convolutional layer used for visualization of trained convolutional network ...

- filters are already learned – no training necessary.



Deconvolutional Layers for Visualization

Problem: Subsampling and pooling in higher layers.

Remember: Placing windows at non-overlapping positions within the feature maps, pooling is accomplished by keeping one activation per window.

Solution: Remember which pixels of a feature map were kept using so called “switch variables”.

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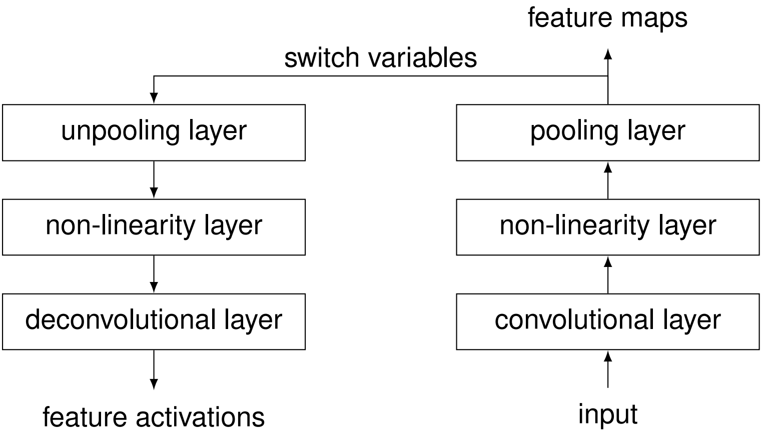


Feature Activations

How does this look?

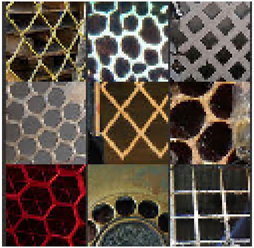
Examples in [ZF13]: Given a validation set, backproject a single activation within a feature map in layer l to analyze which structure excites this particular feature map.

Layer 1: Filters represent Gabor-like filters (for edge detection).

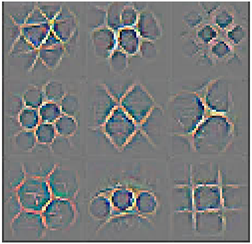
Layer 2: Filters for corners.

Layers above layer 2 are interesting ...

Feature Activations (cont'd)



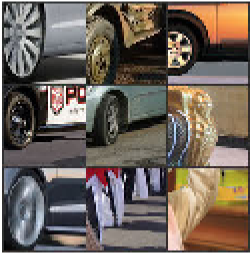
(a) Images.



(b) Activations.

Figure: Activations of **layer 3** backprojected to pixel level [ZF13].

Feature Activations (cont'd)



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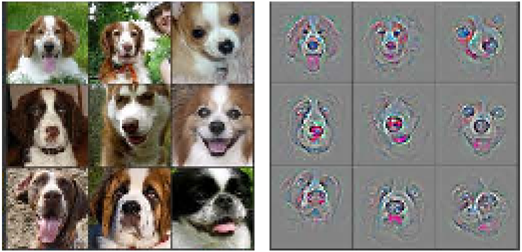
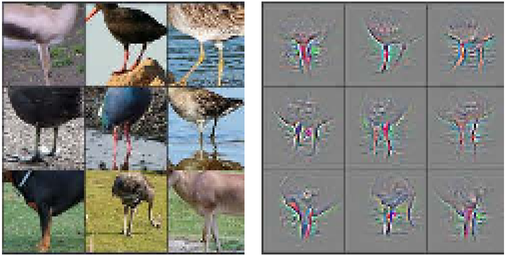


(a) Images. (b) Activations.

Figure: Activations of **layer 4** backprojected to pixel level [ZF13].

Feature Activations (cont'd)



(a) Images. (b) Activations.

Figure: Activations of **layer 4** backprojected to pixel level [ZF13].

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Conclusion

Convolutional networks perform well in computer vision tasks as they learn a feature hierarchy.

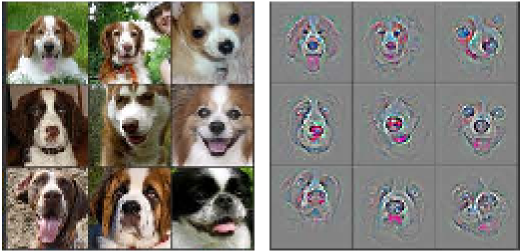
Internal workings of convolutional networks are not well understood.

- [ZF13] use *de*convolutional networks to visualize feature activations;
- this allows to analyze the feature hierarchy and to increase performance.
  - For example by adjusting the filter size and subsampling scheme.

The End

Thanks for your attention!

Paper available at `http://davidstutz.de/seminar-paper-understanding-convolutional-neural-networks/`



Questions?

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